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On Old and New Jacobi Forms

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Certain "index shifting operators" for local and global representations of the Jacobi group are introduced. They turn out to be the representation theoretic analogues of the Hecke operators U_d and V_d on classical Jacobi forms, which

the correspondence between Jacobi forms and elliptic modular forms, this provides some support for a purely local conjecture about the dimension of spaces of spherical vectors in representations of the $\mathfrak{p}$-adic Jacobi group.

INTRODUCTION

While laying the foundations for the representation theory of the Jacobi group G^J in [1], it has always been the policy to consider only representations of a fixed central character. For instance, both in a local or global context, all irreducible representations π of G^J with a fixed central character, indexed by a number m, are in bijection with the irreducible representations $\tilde{\pi}$ of the metaplectic group via the fundamental relation

$$\pi = \tilde{\pi} \otimes \pi_{SW}^m.$$

Here π_{SW}^m is the *Schrödinger–Weil representation* of G^J. We refer to [1, Chap. 2] for the fundamentals of this theory.

On the other hand, on p. 41 of Eichler and Zagier [6] one can find the definition of two Hecke operators changing the index of classical Jacobi forms:

$$U_d\colon J_{k,m} \mapsto J_{k,md^2}, \qquad V_d\colon J_{k,m} \mapsto J_{k,md}. \tag{1}$$

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They are the analogues of the maps $F(z) \mapsto F(dz)$ for elliptic modular forms F, and are therefore the basis for the theory of Jacobi old- and newforms. For elliptic modular forms, the theory of old- and newforms is completely unvisible on the level of representations: The modular forms $F(z)$ and $F(dz)$ generate the same automorphic GL(2)-representation (provided F is an eigenform). However, the analogous statement for Jacobi forms is not true, since the central character of the resulting G^J-representation is directly connected with the index of a Jacobi eigenform. This indicates that the operators U_d and V_d above should at least partly correspond to manipulations on representations. The purpose of this paper is to demonstrate how this can be accomplished.

In the first section we introduce the Jacobi group as a subgroup of GSp(4). The obvious fact that the normalizer of G^J is more than G^J itself yields some automorphisms of the Jacobi group affecting the center. This simple observation leads to the definition of operators U_s and V_s defined on equivalence classes of representations. Since they affect the index (i.e., the central character) of a representation, we call them *index shifting operators*. In Section 2 we examine the effect of these operators on the various classes of local representations (principal series representations, special representations, ...). It turns out that for U_s we have the very simple description

$$U_s(\tilde{\pi} \otimes \pi^m_{SW}) = \tilde{\pi} \otimes \pi^{ms^2}_{SW},$$

while V_s is a little bit more complicated. Nevertheless, $U_s = V_s^2$.

Since we are interested in a group theoretic "explanation" of the classical operators (1), we are led to examine the effect of the index shifting operators on spherical representations. Section 3 takes first steps in this direction, but some questions remain open.

In Section 4 we define global index shifting operators, which are compatible with the local ones, and describe their basic properties. Section 5 contains our main result (Theorem 5.1):

The index shifting operators are compatible with the classical operators (1).

The proof is based on a strong multiplicity-one result for the Jacobi group. Using this fact, Section 6 presents some more detailed remarks on the relation between classical Jacobi forms and automorphic G^J-representations. In particular, classical dimension formulas lead us to make a conjecture about the dimension of the space of spherical vectors in local G^J-representations. Since the classical formulas are not elementary and are only obtained with the help of a trace formula, it might finally turn out difficult to prove the stated conjecture.

But there is some more evidence for this conjecture coming from considering the "certain space" of classical modular forms in the title of Skoruppa and Zagier [13]. We investigate this space and its local analogues more closely in Section 7. We define local and global analogues for elliptic modular forms of the index shifting operators, and with their help state a structure theorem for the "certain space," which matches perfectly the corresponding statement for Jacobi forms (see Theorem 7.6 and formula (7)).

The last section is devoted to give an application of the index shifting operators by determining the local components of automorphic representations of the Jacobi group attached to Jacobi forms of square free index.

We are assuming some familiarity with the representation theory of the Jacobi group, in particular the classification of local representations. A detailed account is given in [1]. Incidentally we shall also make use of the results of [10, 11].

1. THE JACOBI GROUP AND SOME RELATED GROUPS

All of the groups appearing in the following are over an arbitrary commutative ring R. The group containing all the other groups we are considering is

$$\mathrm{GSp}(4) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(4) : \exists x \in \mathrm{GL}(1) \text{ s.t. } AD^t - BC^t = x\mathbf{1}, \right.$$
$$\left. AB^t = BA^t,\ CD^t = DC^t \right\}.$$

GSp(4) has a three-dimensional maximal torus consisting of the matrices $\mathrm{diag}(a, b, c, d)$ with $ac = bd$. We define the one-dimensional subtori

$$T_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a^{-1} \end{pmatrix} : a \in \mathrm{GL}(1) \right\},$$

$$T_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} : a \in \mathrm{GL}(1) \right\}.$$

We will consider SL(2) as a subgroup of GSp(4) via the embedding

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let Q be the standard parabolic subgroup of GSp(4) consisting of matrices whose bottom row is $(0, 0, 0, *)$. Then Q is a proper maximal parabolic subgroup containing T_1, T_2, and SL(2). The unipotent radical of Q is the Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} : \lambda, \mu, \kappa \in R \right\};$$

the elements of H will often be abbreviated as (λ, μ, κ). The parabolic Q is a semidirect product

$$Q = T_1 T_2 \, \mathrm{SL}(2) \ltimes H.$$

The Jacobi group is by definition the subgroup

$$G^J = \mathrm{SL}(2) \ltimes H$$

of Q. We will also have reasons to consider the extended Jacobi groups

$$G_1^J = T_1 \ltimes G^J, \qquad G_2^J = T_2 \ltimes G^J.$$

For $x, y \in \mathrm{GL}(1)$ identify x with the element $\mathrm{diag}(1, x, 1, x^{-1})$, of T_1, and y with the element $\mathrm{diag}(1, 1, y^{-1}, y^{-1})$ of T_2. Then the action of T_1 resp. T_2 on G^J by conjugation is given by

$$x \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\lambda, \mu, \kappa)\, x^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (x\lambda, x\mu, x^2\kappa),$$

$$y \begin{pmatrix} a & b \\ c & c \end{pmatrix} (\lambda, \mu, \kappa)\, y^{-1} = \begin{pmatrix} a & yb \\ y^{-1}c & d \end{pmatrix} (\lambda, y\mu, y\kappa).$$

The fact that G^J is normalized by the tori T_j thus produces non-trivial automorphisms

$$U_x \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\lambda, \mu, \kappa) \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} (x\lambda, x\mu, x^2\kappa) \tag{2}$$

and

$$V_y\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix}(\lambda, \mu, \kappa) \mapsto \begin{pmatrix} a & yb \\ y^{-1}c & d \end{pmatrix}(\lambda, y\mu, y\kappa) \tag{3}$$

of G^J, for any $x, y \in R^*$. These automorphisms affect the center $Z = (0, 0, *)$ of G^J (which coincides with the center of H). We will see that they have a lot to do with the operators U and V appearing in [6].

2. LOCAL INDEX SHIFTING

Now let $R = F$ be a local field of characteristic zero. We fix an element $s \in F^*$ and consider the automorphisms U_s, V_s of G^J of the last section. We define operators of the same name acting on the set of (equivalence classes of) irreducible, admissible representations of G^J with non-trivial central character. If π is such a representation, then let

$$U_s\pi := \pi \circ U_s, \qquad V_s\pi := \pi \circ V_s.$$

Let ψ be a fixed non-trivial additive character of F. Then the central character of π is ψ^m for some uniquely determined $m \in F^*$, where $\psi^m(x) := \psi(mx)$. With the classical theory of Jacobi forms in mind, we also say that π has *index* m. It is clear by the formulas (2) and (3) that if π is of index m, then $U_s\pi$ (resp. $V_s\pi$) is of index ms^2 (resp. ms). For this reason we say that U_s and V_s are *index shifting operators.*

We want to know explicitly the effect of these operators on irreducible representations. We first have a look at what happens to the Schrödinger and Weil representations. For these representations and the fundamental role they play in the representation theory of the Jacobi group, see [1] 2.5, 2.6.

2.1. PROPOSITION. *For any $m, s \in F^*$ we have*

$$U_s\pi_S^m = \pi_S^{ms^2}, \qquad U_s\pi_{SW}^m = \pi_{SW}^{ms^2}, \qquad U_s\pi_W^{m\pm} = \pi_W^{ms^2\pm},$$
$$V_s\pi_S^m = \pi_S^{ms}, \qquad V_s\pi_{SW}^m = \pi_{SW}^{ms}, \qquad V_s\pi_W^{m\pm} = \pi_W^{ms\pm}.$$

Proof. This can be seen by giving explicit isomorphisms on the Schrödinger models of these representations: $f \mapsto \tilde{f}$ with $\tilde{f}(x) = f(sx)$ is a vector space automorphism of the Schwartz space $\mathscr{S}(F)$ which intertwines $U_s\pi_{SW}^m$ and $\pi_{SW}^{ms^2}$. This is proved by a simple calculation using the well-known explicit formulas for the Schrödinger–Weil representation (see [1, 2.1, 2.5]). Similarly, the identity intertwines $V_s\pi_{SW}^m$ and π_{SW}^{ms}. These explicit maps make the assertion about the representations $\pi_W^{m\pm}$ obvious. ▮

We recall from [1, 2.6] that every irreducible, admissible representation π^J of G^J is of the form $\pi^J = \tilde{\pi} \otimes \pi_{SW}^m$ with an irreducible, admissible representation $\tilde{\pi}$ of the metaplectic group Mp.

2.2. Proposition. *Assume $\pi^J =: \tilde{\pi} \otimes \pi_{SW}^m$ is an irreducible, admissible representation of G^J. Then*

$$U_s \pi^J = \tilde{\pi} \otimes \pi_{SW}^{ms^2}.$$

In particular, for $F = \mathbb{R}$ we have

$$U_s \pi_{m,t,v}^J = \pi_{ms^2,t,v}^J \qquad (v \in \{-1/2, 1/2\},\ t \in \mathbb{C} \backslash (\mathbb{Z} + 1/2)),$$

$$U_s \pi_{m,k}^{J\pm} = \pi_{ms^2,k}^{J\pm} \qquad (k \geqslant 1).$$

If F is non-archimedean, then U_s maps supercuspidals to supercuspidals, and

$$U_s \pi_{\chi,m}^J = \pi_{\chi,ms^2}^J, \qquad U_s \sigma_{\xi,m}^J = \sigma_{\xi,ms^2}^J, \qquad U_s \sigma_{\xi,m}^{J\pm} = \sigma_{\xi,ms^2}^{J\pm}.$$

Proof. The first assertion simply follows from formula (2), showing that the SL(2)-part of G^J is unaffected by the automorphism U_s. The assertions in the case $F = \mathbb{R}$ are then trivial, because $\pi_{m,t,v}^J = \tilde{\pi}_{t,v} \otimes \pi_{SW}^m$, and similarly for the discrete series representations. In the non-archimedean case the assertions are slightly less trivial, because with our notations the metaplectic part of, say, the principal series representations depends on the index:

$$\pi_{\chi,m}^J = \tilde{\pi}_{\chi,-m} \otimes \pi_{SW}^m.$$

So we have to use the fact that

$$\tilde{\pi}_{\chi,-m} = \tilde{\pi}_{\chi,-ms^2},$$

and similarly for the special and Weil representations (see also the previous proposition). Since U_s permutes all the (equivalence classes of) irreducible, admissible representations of G^J, it must map supercuspidals to supercuspidals. ▮

2.3. Proposition. *For $F = \mathbb{R}$ we have*

$$V_s \pi_{m,t,v}^J = \pi_{ms,t,v}^J \qquad (v \in \{-1/2, 1/2\},\ t \in \mathbb{C} \backslash (\mathbb{Z} + 1/2)),$$

$$V_s \pi_{m,k}^{J\pm} = \pi_{ms,k}^{J\pm} \qquad (k \geqslant 1).$$

If F is non-archimedean, then V_s maps supercuspidals to supercuspidals, and

$$V_s \pi^J_{\chi, m} = \pi^J_{\chi, ms}, \qquad V_s \sigma^J_{\xi, m} = \sigma^J_{\xi, ms}, \qquad V_s \sigma^{J\pm}_{\xi, m} = \sigma^{J\pm}_{\xi, ms}.$$

Proof. We first treat the non-archimedean case. For π to be supercuspidal means that for any vector v in the space of π we have

$$\int_{\mathfrak{p}^{-l}} \int_{\mathfrak{p}^{-n}} \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} (0, \mu, 0)\right) v \, dx \, d\mu = 0$$

for any large enough $l, n \in \mathbb{N}$ ($\mathfrak{p}$ is the maximal ideal of $\mathcal{O}$, the ring of integers of F). This property is clearly preserved by V_s. (Of course, we could also have used the argument of the previous proposition.) Now let $\pi = \pi^J_{\chi, m}$ be a principal series representation. It can be realized on the space $\mathcal{B}^J_{\chi, m}$ of smooth functions $f\colon G^J \to \mathbb{C}$ which transform as

$$f\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} (0, \mu, \kappa)\, g\right)$$
$$= \psi^m(\kappa)\, \xi(a)\, |a|^{3/2} f(g) \qquad \text{for} \quad a \in F^*, \quad x, \mu, \kappa \in F, \quad g \in G^J$$

(see [1, 5.4]). The action of G^J on this space is by right translation ρ. A quick calculation shows that the map $f \mapsto f \circ V_s$ establishes an isomorphism

$$\mathcal{B}_{\chi, m} \xrightarrow{\sim} \mathcal{B}_{\chi, ms}.$$

In fact, this is an intertwining map for $\rho \circ V_s$ on the left hand side, and right translation on the right hand side. This proves that $V_s \pi^J_{\chi, m} = \pi^J_{\chi, ms}$. A similar argument is applicable to the special representations. The assertion about the Weil representations follows from Proposition 2.1.

The case $F = \mathbb{R}$ can be handled similarly to the non-archimedean principal series representations, since by [1, 3.3] the real representations of G^J can also be obtained as induced representations. ∎

We stress the fact that, in contrast to the operator U_s, it is *not* true in general that $V_s \pi = \tilde{\pi} \otimes \pi^{ms}_{SW}$ if $\pi = \tilde{\pi} \otimes \pi^m_{SW}$. The proposition shows that it *is* true for the real representations, but not, for example, for the non-archimedean principal series representations: The representation

$$\pi^J_{\chi, m} = \tilde{\pi}_{\chi, -m} \otimes \pi^m_{SW}$$

is sent to

$$\pi^J_{\chi, ms} = \tilde{\pi}_{\chi, -ms} \otimes \pi^{ms}_{SW},$$

and $\tilde{\pi}_{\chi,\,-ms}$ is different from $\tilde{\pi}_{\chi,\,-m}$. In fact, directly from the definition in [14] (or [1, 5.3]) we have

$$\tilde{\pi}_{\chi,\,-ms} = \tilde{\pi}_{\chi\chi_s,\,-m},$$

where χ_s denotes the quadratic character

$$\chi_s(x) = (x, s) \qquad (x \in F^*)$$

(Hilbert symbol).

2.4. Proposition. *As operators on representations we have* $U_s = V_s^2$.

Proof. By formulas (2) and (3) we have

$$\begin{aligned} V_s^2\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\lambda, \mu, \kappa)\right) &= \begin{pmatrix} a & s^2 b \\ s^{-2} c & d \end{pmatrix}(\lambda, s^2\mu, s^2\kappa) \\ &= \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}(s\lambda, s\mu, s^2\kappa)\begin{pmatrix} s^{-1} & 0 \\ 0 & s \end{pmatrix}, \end{aligned}$$

showing that $V_s^2(g)$ and $U_s(g)$ are conjugate by the matrix $\left(\begin{smallmatrix} s & 0 \\ 0 & s^{-1} \end{smallmatrix}\right)$, for every $g \in G^J$. The assertion follows. ∎

For later use, we note the following lemma. The function δ appearing here is the *Weil character*, cf. [1, 5.3].

2.5. Lemma. *Let* $\pi = \tilde{\pi} \otimes \pi_{SW}^m$ *be an irreducible, admissible representation of* G^J, *and let* $V_s\pi = \tilde{\pi}' \otimes \pi_{SW}^{ms}$, *with the corresponding representations* $\tilde{\pi}$ *resp.* $\tilde{\pi}'$ *of the metaplectic group. Let* λ *resp.* λ' *be the central characters of* $\tilde{\pi}$ *resp.* $\tilde{\pi}'$. *Then*

$$\lambda(-\mathbf{1})\,\delta_m(-\mathbf{1}) = \lambda'(-\mathbf{1})\,\delta_{ms}(-\mathbf{1}).$$

Proof. Let W be a space for $\tilde{\pi}$, regarded as a projective representation of SL(2), and let W' be a space for $\tilde{\pi}'$. By definition of V_s there exists an isomorphism

$$\varphi\colon W \xrightarrow{\sim} W',$$

which intertwines the (projective) SL(2)-actions $\tilde{\pi} \circ V_s$ on W and $\tilde{\pi}'$ on W'. Let

$$\psi\colon \mathcal{S}(F) \xrightarrow{\sim} \mathcal{S}(F)$$

be the isomorphism $f \mapsto (x \mapsto f(xs))$ which was already mentioned in the proof of Proposition 2.1; it intertwines the (projective) G^J-actions $\pi_{SW}^m \circ V_s$

on the left hand side, and π^{ms}_{SW} on the right hand side. Now consider the isomorphism

$$\varphi \otimes \psi \colon W \otimes \mathscr{S}(F) \xrightarrow{\sim} W' \otimes \mathscr{S}(F),$$

which by construction is an intertwining map for $V_s\pi$ and π'. Let f be an even Schwartz function, and let w be any non-zero element of W. Since ψ leaves the space $\mathscr{S}(F)^+$ invariant, we compute

$$\begin{aligned}(\varphi \otimes \psi)((V_s\pi)(-\mathbf{1})(w \otimes f)) &= \pi'(-\mathbf{1})(\varphi \otimes \psi(w \otimes f))\\ &= \pi'(-\mathbf{1})(\varphi(w) \otimes \psi(f))\\ &= (\tilde{\pi}(-\mathbf{1})\,\varphi(w)) \otimes (\pi^{ms}_{W}(-\mathbf{1})\,\psi(f))\\ &= \lambda'(-\mathbf{1})\,\delta_{ms}(-1)\,\varphi(w) \otimes \psi(f),\end{aligned}$$

by the explicit formulas for the Weil representation. On the other hand,

$$\begin{aligned}(\varphi \otimes \psi)((V_s\pi)(-\mathbf{1})(w \otimes f)) &= (\varphi \otimes \psi)(\pi(-\mathbf{1})(w \otimes f))\\ &= (\varphi \otimes \psi)(\lambda(-\mathbf{1})\,\delta_m(-1)\,w \otimes f)\\ &= \lambda(-\mathbf{1})\,\delta_m(-1)\,\varphi(w) \otimes \psi(f).\end{aligned}$$

The assertion follows. ∎

We recall that in the non-archimedean case a local representation of G^J is called *spherical* if it contains a non-zero vector invariant under $K^J = G^J(\mathscr{O})$. The so-called Heisenberg involutions act on the space of spherical vectors by id or $-$id [1, Proposition 7.5.3]).

2.6. Proposition. *Consider the non-archimedean case. Assume that both π and $V_s\pi$ are spherical G^J-representations. Then the spherical vectors in both representations have the same eigenvalue under the respective Heisenberg involutions.*

Proof. Let $\varepsilon \in \{\pm 1\}$ be the eigenvalue under the Heisenberg involution of a spherical vector in π. Writing $\pi = \tilde{\pi} \otimes \pi^m_{SW}$ as usual, let λ be the central character of the metaplectic representation $\tilde{\pi}$. Then by [1, 7.5.3] the relation

$$\varepsilon = \lambda(-\mathbf{1})\,\delta_m(-1)$$

holds. An analogous formula is valid for $V_s\pi$. Thus our assertion follows from the previous lemma. ∎

3. HECKE ALGEBRA AUTOMORPHISMS AND SPHERICAL VECTORS

Let G be a $\mathfrak{p}$-adic group and K an open-compact subgroup. The *Hecke algebra* $\mathscr{H}(G, K)$ of the pair (G, K) is the space of compactly supported left and right K-invariant complex-valued functions on G, endowed with the multiplication

$$(f * g)(x) = \int_G f(xy)\, g(y^{-1})\, dy \qquad (f, g \in \mathscr{H}(G, K)).$$

Let $V\colon G \to G$ be an automorphism which induces an automorphism $K \to K$. A routine argument shows that

$$\int_G f(V(x))\, dx = \int_G f(x)\, dx$$

for all locally constant and compactly supported functions f on G. Using this fact, a simple calculation shows that the map

$$\mathscr{H}(G, K) \to \mathscr{H}(G, K)$$
$$f \mapsto f \circ V$$

is an isomorphism of $\mathbb{C}$-algebras.

Now let F be a $\mathfrak{p}$-adic field with ring of integers $\mathcal{O}$. We specialize to the case $G = G^J(F)$, $K = K^J = G^J(\mathcal{O})$, and $V = V_s$ with s a unit in $\mathcal{O}$. It is trivial by formula (3) that V_s induces an automorphism of K^J. Hence we obtain an automorphism.

$$V_s\colon \mathscr{H}(G^J, K^J) \xrightarrow{\sim} \mathscr{H}(G^J, K^J) \tag{4}$$

of the Jacobi Hecke algebra, denoted also by V_s. When considering representations of G^J with a fixed index m, the relevant Hecke algebra is not $\mathscr{H}(G^J, K^J)$ itself, but the algebra $\mathscr{H}(G^J, K^J)_m$ consisting of left and right K^J-invariant functions f which are compactly supported modulo the center Z of G^J, and which satisfy $f(xz) = \psi^{-m}(z)\, f(x)$ for all $x \in G^J$ and $z \in Z \simeq F$ (the additive character ψ is fixed throughout). As is not hard to show, the map

$$\Xi_m\colon \mathscr{H}(G^J, K^J) \to \mathscr{H}(G^J, K^J)_m,$$
$$f \mapsto \left(x \mapsto \int_Z f(xz)\, \psi^m(z)\, dz\right),$$

is a surjective algebra homomorphism. If f is in the kernel of Ξ_m, it is easy to see that $f \circ V_s$ is in the kernel of Ξ_{ms}. Hence the automorphism (4) identifies $\ker(\Xi_m)$ with $\ker(\Xi_{ms})$. Passage to the quotient yields the following result.

3.1. PROPOSITION. *If $s \in \mathcal{O}^*$, then the automorphism V_s induces an automorphism of Hecke algebras*

$$\mathscr{H}(G^J, K^J)_m \xrightarrow{\sim} \mathscr{H}(G^J, K^J)_{ms}.$$

In [5, 9] the following results about the structure of $\mathscr{H}(G^J, K^J)_m$ were proved. We assume ψ has conductor $\mathcal{O}$. Let v be the normalized valuation on F. In the *good case* $v(m) = 0$ we have

$$\mathscr{H}(G^J, K^J)_m \cong \mathbb{C}[X^{\pm 1}]^W,$$

and in the *almost good case* $v(m) = 1$ we have

$$\mathscr{H}(G^J, K^J)_m \cong \mathbb{C}[X^{\pm 1}]^W \times \mathbb{C}.$$

Here $\mathbb{C}[X^{\pm 1}]^W$ means those polynomials which are invariant under the Weyl group action $X \mapsto X^{-1}$. (The result in the good case was first obtained by T. Shintani, even for higher degree Jacobi groups. See Murase [8, Sect. 5].)

Our Proposition 3.1 now explains why the structure of $\mathscr{H}(G^J, K^J)_m$ really only depends on the valuation of m, a fact which one might wonder about while reading the above mentioned papers. It is true that while this valuation increases, the structure becomes more and more complicated. In fact, if $v(m) \geqslant 2$, which is called the *bad cases*, the Hecke algebra $\mathscr{H}(G^J, K^J)_m$ is no longer commutative.

The Hecke algebra isomorphisms we constructed in Proposition 3.1 simply reflect the fact that if v is a spherical vector (meaning non-zero and K^J-invariant) in some representation π of G^J, then the same v is also a spherical vector for the representation $V_s \pi = \pi \circ V_s$ (we are still assuming $s \in \mathcal{O}^*$). Now we examine what happens if s is no longer assumed to be a unit.

Thus let $0 \neq s \in \mathcal{O}$ be arbitrary. From (2) the following observation is trivial:

$$\text{if } \pi \text{ is spherical, and } 0 \neq s \in \mathcal{O}, \text{ then } U_s \pi \text{ is spherical.} \tag{5}$$

Assume $v(s)$ is even and non-negative. Then we can write $s = u\omega^{2l}$ where $u \in \mathcal{O}^*$ and where $\omega \in \mathcal{O}$ is a prime element (i.e. $v(\omega) = 1$). By Proposition 2.4 we then have

$$V_s = V_u \circ U_\omega^l.$$

Since V_u leaves the property of being spherical unaffected, it follows from (5) that

if π is spherical, and $0 \neq s \in \mathcal{O}$ with $v(s)$ even, then $V_s\pi$ is spherical.

It seems much more difficult to decide if $V_\omega \pi$ is spherical, provided π is spherical. In general this is not true: By [10, Theorem 3.3.1], the positive Weil representation $\pi = \sigma^{J+}_{\xi, m}$ with $\xi \in \mathcal{O}^* \backslash \mathcal{O}^{*2}$ and $v(m) = 0$ is spherical, while its image $V_\omega \pi = \sigma^{J+}_{\xi, m\omega}$ is not spherical. However, it is also shown in [10] that the principal series representations $\pi = \pi^J_{\chi, m}$ with $v(m) = 0$ and unramified χ are spherical, as well as their images $V_\omega \pi = \pi^J_{\chi, m\omega}$. We thus make the following conjecture.

Conjecture. If π is a spherical principal series representation, and $0 \neq s \in \mathcal{O}$, then $V_s\pi$ is also spherical.

4. GLOBAL INDEX SHIFTING

Now assume that F is a global field with adele ring $\mathbb{A}$. We fix a non-trivial character ψ of $\mathbb{A}$, trivial on F. Any such character is then of the form ψ^m for a uniquely determined $m \in F^*$. If a global representation π of $G^J = G^J(\mathbb{A})$ has central character ψ^m, we also say that π has *index* m.

Now if s is any idele, we can introduce automorphisms U_s and V_s of $G^J(\mathbb{A})$ given by the formulas (2) and (3). Exactly as in the local case we define the *index shifting operators* U_s and V_s on global representations π of G^J by

$$U_s \pi := \pi \circ U_s, \qquad V_s \pi = \pi \circ V_s.$$

Since the global representations we have in mind are all *automorphic*, we assume $s \in F^*$ from now on. It is clear that if π is of index m, then $U_s\pi$ (resp. $V_s\pi$) is of index ms^2 (resp. ms).

4.1. PROPOSITION. *Assume $s \in F^*$. Then U_s and V_s take (cuspidal) automorphic representations to (cuspidal) automorphic representations. If $\hat{G}^J_m$ denotes the set of (equivalence classes of) automorphic representations of G^J with index m, then U_s and V_s induce bijections*

$$U_s : \hat{G}^J_m \to \hat{G}^J_{ms^2}, \qquad V_s : \hat{G}^J_m \to \hat{G}^J_{ms}.$$

U_s and V_s are compatible with their local versions defined in the previous section:

if $\quad \pi = \otimes \pi_{\mathfrak{p}}, \qquad$ *then* $\quad U_s \pi = \otimes (U_{s,\mathfrak{p}} \pi_{\mathfrak{p}}) \quad$ *and* $\quad V_s \pi = \otimes (V_{s,\mathfrak{p}} \pi_{\mathfrak{p}}).$

The operator U_s *is simply the map*

$$\pi = \tilde{\pi} \otimes \pi_{SW}^{m} \mapsto \tilde{\pi} \otimes \pi_{SW}^{ms^2},$$

while V_s *is definitely not of this simple form. We have* $U_s = V_s^2$.

Proof. Suppose π is an automorphic representation of G^J of index m. Let W be the space of automorphic forms on G^J realizing π. If we associate with any $f \in W$ the function $f \circ U_s$, then we get a new space W' of automorphic forms on G^J. The map $f \mapsto f \circ U_s$ certainly is an isomorphism

$$W \xrightarrow{\sim} W',$$

and takes cuspidal functions to cuspidal functions. This isomorphism is compatible with the actions of G^J on both sides, where we let G^J act on W' by right translation, and on W by right translation composed with U_s (i.e., by $U_s\pi$). This proves that $U_s\pi$ is again automorphic, and the asserted bijection is evident. The assertions about V_s are proved analogously.

The compatibility of the local and global index shifting operators is obvious. Using this, the remaining assertions follow from the remarks made after the proof of Proposition 2.3, and from Proposition 2.4. ∎

Let $\hat{G}^J$ be the disjoint union of all the sets $\hat{G}^J_m$ with $m \in F^*$, i.e., $\hat{G}^J$ is the set of all (equivalence classes of) automorphic representations of G^J with non-trivial central character. Let $\widehat{\mathrm{PGL}}(2)$ be the set of (equivalence classes of) automorphic representations of PGL(2) (everything is over our global field F).

It was shown in [11] that there is a canonical lifting map

$$\hat{G}^J \to \widehat{\mathrm{PGL}}(2), \tag{6}$$

i.e., a correspondence which also has a local definition, and the local and global maps are compatible. The lift (6) is obtained as follows. For $\pi^J \in \hat{G}^J$, there is a unique $m \in F^*$ such that $\pi^J \in \hat{G}^J_m$, namely, m is the index of π^J. Then there is a unique automorphic representation $\tilde{\pi}$ of the metaplectic group Mp such that

$$\pi^J = \tilde{\pi} \otimes \pi_{SW}^{m}.$$

The image of π^J under the lift (6) is then defined as the image of $\tilde{\pi}$ under the mth Waldspurger correspondence, by which we mean the correspondence between automorphic representations of Mp and of PGL(2) described in [14, 15], where the underlying character is ψ^m.

4.2. Proposition. *For any* $\pi^J \in \hat{G}^J$ *and* $s \in F^*$, *the representations* π^J, $U_s\pi^J$ *have the same image under the lift* (6).

Proof. Since $U_s = V_s^2$, it is enough to prove this for V_s. Let $\pi^J = \otimes \pi^J_{\mathfrak{p}}$ be the decomposition of $\pi^J \in \hat{G}^J$ in local components, and let $\pi = \otimes \pi_{\mathfrak{p}}$ be the decomposition of the image of π^J under the Jacobi-PGL(2)-lift. It is known that for almost every finite place $\mathfrak{p}$, the local component $\pi^J_{\mathfrak{p}}$ is a principal series representation $\pi^J_{\chi, m}$, for some character χ of $F^*_{\mathfrak{p}}$. By the properties of the local Jacobi-PGL(2)-lift (see [11]), we have $\pi_{\mathfrak{p}} = \pi(\chi, \chi^{-1})$, the well-known principal series representation for GL(2). Now, by Proposition 2.3, at the same place $\mathfrak{p}$ the representation $V_s \pi^J$ has local component $\pi^J_{\chi, ms}$, which is also mapped to $\pi(\chi, \chi^{-1})$ under the local lift. Thus our assertion follows by strong multiplicity one for GL(2). ∎

In other words, the operators U_s and V_s respect the fibres of the lift (6). An L-packet of G^J is defined to be the set of all automorphic representations of G^J of a fixed index sharing the same image under the Jacobi-PGL(2)-lift. We can take over Waldspurger's results [15] and conclude that L-packets of G^J are finite, but may contain more than one element.

4.3. Corollary. *The bijections* $U_s\colon \hat{G}^J_m \to \hat{G}^J_{ms^2}$ *and* $V_s\colon \hat{G}^J_m \to \hat{G}^J_{ms}$ *induce bijections of L-packets.*

5. INDEX SHIFTING AND CLASSICAL JACOBI FORMS

It was shown in the last chapter of [1] that there is a correspondence between classical Jacobi forms on $\mathscr{H} \times \mathbb{C}$ and automorphic representations of the Jacobi group. We shall now show that under this correspondence our global index shifting operators correspond to the operators U and V on classical Jacobi forms defined in [6]. We first recall from [1] what is needed from the correspondence between Jacobi forms and representations. Our attention will mainly be restricted to cusp forms.

On the space $J_{k,m}$ of Jacobi forms of weight k and index m we have the following Hecke operators. For any prime number $p \nmid m$ there is an operator $T_{EZ}(p)$, defined on p. 41 of [6]. Moreover, for any prime number $p \mid m$ there is an operator W_p, defined on p. 60 of [6], which is an involution. An element of $J_{k,m}$ is called an *eigenform* if it is a simultaneous eigenvector for all $T_{EZ}(p)$ and W_p. Since all of these operators commute, there exists a basis of $J_{k,m}$ consisting of eigenforms.

Now let f be a cuspidal eigenform. In a straightforward manner we can associate to f a function Φ_f on the group $G^J(\mathbb{A})$, where $\mathbb{A}$ denotes the adeles of $\mathbb{Q}$. This Φ_f is an automorphic form on G^J and lies in a canonically defined Hilbert space

$$L^2_0(G^J(\mathbb{Q})\backslash G^J(\mathbb{A}))_m.$$

The Jacobi group $G^J(\mathbb{A})$ acts on this space by right translation. Let π_f be the subrepresentation of $L_0^2(G^J(\mathbb{Q})\backslash G^J(\mathbb{A}))_m$ generated by Φ_f. Then it is a consequence of a strong multiplicity one result for the Jacobi group that π_f is irreducible (cf. [1, 7.5]). π_f is the automorphic representation associated with the classical Jacobi form f. Since f has index m, the representation π_f will also have index m, in the sense of the previous section, provided that for ψ the global "standard character" is chosen (which is characterized by the property that it is trivial on every $\mathbb{Z}_p$ and takes the value $e^{2\pi i x}$ on $x \in \mathbb{R}$). In the following, whenever the underlying number field is $\mathbb{Q}$, we agree to choose this standard character.

5.1. THEOREM. *Let $\mathbb{Q}$ be the underlying global field. For any natural number s, our index shifting operators U_s and V_s on automorphic representations are compatible with the classical operators U_s, V_s defined on Jacobi forms, in the sense that the following diagrams are commutative*:

$$\begin{array}{ccc} \hat{G}^J_m & \xrightarrow{U_s} & \hat{G}^J_{ms^2} \\ \uparrow & & \uparrow \\ J^{\mathrm{cusp}}_{k,m}-\mathrm{EF} & \xrightarrow{U_s} & J^{\mathrm{cusp}}_{k,ms^2}-\mathrm{EF} \end{array} \qquad \begin{array}{ccc} \hat{G}^J_m & \xrightarrow{V_s} & \hat{G}^J_{ms} \\ \uparrow & & \uparrow \\ J^{\mathrm{cusp}}_{k,m}-\mathrm{EF} & \xrightarrow{V_s} & J^{\mathrm{cusp}}_{k,ms}-\mathrm{EF} \end{array}$$

Here $J^{\mathrm{cusp}}_{k,m}$-EF stands for the set of eigenforms in $J^{\mathrm{cusp}}_{k,m}$

Proof. We prove this for V_s, the other case being treated similarly. Let $f \in J^{\mathrm{cusp}}_{k,m}$ be an eigenform and $\pi_f = \otimes \pi_q$ the associated representation, where q runs over the places of $\mathbb{Q}$. Similarly, let $\pi' = \otimes \pi'_q$ be the representation generated by $f \mid V_s$. We have to show $V_s \pi = \pi'$. By Proposition 2.3 this is true at the archimedean place, since $\pi_\infty = \pi^J_{m,k}$ and $\pi'_\infty = \pi^J_{ms,k}$ by [1, 7.5.5].

As for the finite places, fix any $q \nmid ms\infty$. Let $c(q)$ be the corresponding Hecke eigenvalue of f:

$$f \mid T_{EZ}(q) = c(q)\, f.$$

By [1, 7.5] this eigenvalue determines the local component π_q. Namely, π_q is a principal series representation $\pi^J_{\chi,m}$ with (the Weyl group orbit of) the unramified character χ of $\mathbb{Q}_q^*$ determined by

$$c(q) = q^{k-3/2}(\chi(q) + \chi(q)^{-1}).$$

Since the classical Hecke operators $T_{EZ}(q)$ and V_s commute, the Jacobi form $f \mid V_s$ also has $T_{EZ}(q)$-eigenvalue $c(q)$. It follows that $\pi'_q = \pi^J_{\chi,ms}$ with the same χ. By Proposition 2.3, it follows that $\pi'_q = V_s \pi_q$. We have proved that π and π' have the same local components at almost all places.

By strong multiplicity one for the metaplectic group (see [1, 7.5]) we are done if we can show that the metaplectic representations corresponding to π and π' have the same central character. Let q be any prime number. If q divides the index of a Jacobi form, we have the classical operator W_q. If q does not divide the index, we let W_q be the identity. Hence we have commuting operators W_q for all q, and f is a simultaneous eigenfunction for all of these. Since the W_q commute with V_s (any q and s), $f \mid V_s$ is also an eigenfunction, with the same eigenvalues. Fix any q, and let $\varepsilon \in \{\pm 1\}$ be the corresponding eigenvalue. By [1, 7.4.8 and 7.5.3], the local Mp-representation corresponding to π_q has central character λ determined by

$$\varepsilon = \lambda(-\mathbf{1})\,\delta_m(-1),$$

and the local Mp-representation corresponding to π'_q has central character λ' determined by

$$\varepsilon = \lambda'(-\mathbf{1})\,\delta_{ms}(-1).$$

In particular, we find

$$\lambda(-\mathbf{1})\,\delta_m(-1) = \lambda'(-\mathbf{1})\,\delta_{ms}(-1).$$

By Lemma 2.5 it follows that λ' is also the central character of the Mp-representation corresponding to $V_s \pi_q$. This is exactly what had to be shown. ∎

In this proof we could not make use of Proposition 2.6, because it is not clear a priori that the representation $V_s \pi_f$ is spherical (meaning at every finite place). However, now that Theorem 5.1 is proved, we know that V_s sends spherical representations to spherical representations, provided $s \in \mathbb{N}$ (and the underlying number field is $\mathbb{Q}$). This leads us to make the following definition. A cuspidal automorphic representation π of G^J over $\mathbb{Q}$ is called *classical* if

- π is spherical at every finite place,
- the index m of π is a natural number (we have fixed the global standard character), and
- the infinite component of π is $\pi^J_{k,m}$ for some integer $k \geqslant 1$, called the weight of π.

It is clear that the classical automorphic representations of G^J are exactly those containing a classical cuspidal Jacobi eigenform (considered as a function on $G^J(\mathbb{A})$). Let us denote the set of classical automorphic representations of G^J with index m and weight k by the symbol

$$\hat{G}^{J,0}_{k,m}.$$

Theorem 5.1 proves that the index shifting operators U_s and V_s for $s \in \mathbb{N}$ induce maps

$$\hat{G}^{J,0}_{k,m} \to \hat{G}^{J,0}_{k,ms^2} \qquad \text{and} \qquad \hat{G}^{J,0}_{k,m} \to \hat{G}^{J,0}_{k,ms}.$$

From what we have said at the end of Section 3, this is trivial for U, but surprising for V_s, since it is not true in general for local representations that if π is spherical, then $V_s\pi$ is also. As we have mentioned in Section 3, a counterexample is the spherical positive Weil representation of $G^J(\mathbb{Q}_p)$

$$\sigma^{J+}_{\xi,m} \qquad \text{with} \quad \xi \in \mathbb{Z}_p^* \backslash \mathbb{Z}_p^{*2}$$

for $p \nmid m$. We can thus conclude:

5.2. COROLLARY. *For any prime number p not dividing m, the spherical representation $\sigma^{J+}_{\xi,m}$ of $G^J(\mathbb{Q}_p)$ with $\xi \in \mathbb{Z}_p^* \backslash \mathbb{Z}_p^{*2}$ does not appear as a local component in any automorphic representation attached to a cuspidal Jacobi eigenform $f \in J_{k,m}$.*

This is a very special case of a theorem of Waldspurger, stating that positive Weil representations do not appear as local components in cuspidal automorphic representations of the metaplectic group (cf. [14, Proposition 23, 1, 7.5.7]).

6. CONJECTURES ABOUT OLDFORMS AND NEWFORMS

In this section we are working mostly over the global field $\mathbb{Q}$. We have proved in Theorem 5.1 that our index shifting operators U_s and V_s are compatible with the classical operators of the same name ($s \in \mathbb{N}$). However, something is lost in changing from classical Jacobi forms to representations, since on the level of representations we have $U_s = V_s^2$, while this is definitely not true for the classical operators. For example, consider a cuspidal Jacobi eigenform $f \in J_{k,1}$ with corresponding representation $\pi = \pi_f$. By the formula

$$J^{\text{cusp, old}}_{k,m} = \bigoplus_{\substack{l,\,l' \in \mathbb{N},\; ll' \neq 1 \\ l^2 l' \mid m}} J^{\text{cusp, new}}_{k,m/l^2l'} \mid U_l V_{l'} \tag{7}$$

(see p. 49 of [6]), the Jacobi forms

$$f \mid V_{p^2} \qquad \text{and} \qquad f \mid U_p,$$

both elements of $J_{k,\,p^2}$ (where p is any prime), are linearly independent. On the other hand, both of them generate the same representation $\pi' := U_p\pi = V_p^2\pi$. Hence the space of Jacobi forms (considered as functions on $G^J(\mathbb{A})$) contained in π' is at least two-dimensional. In fact, this can be seen locally: The local components of π' at finite places $q \neq p$ are spherical principal series representations in the good case, which have one-dimensional space of spherical vectors, while the p-component equals $\pi^J_{\chi,\,p^2}$ for some unramified character χ of $\mathbb{Q}_p^*$, and this representations has a space of spherical vectors which is at least two-dimensional by [9, 1.3.9]. Actually, we now see that it must be exactly two-dimensional, because otherwise we had a contradiction to formula (7).

The situation for G^J is thus different from the one for GL(2). Consider a classical modular form $f \in \mathcal{S}_k(\Gamma_0(m))$, assumed to be a newform and a Hecke eigenform. Let π be the associated representation of GL(2). For any integer $s \geqslant 2$, the function $z \mapsto f(sz)$ is an old eigenform of level ms. But it also lies in the space of π, and hence the GL(2)-representation associated with this oldform is again π. In other words, the oldforms are not visible on the level of representations. For the Jacobi group, they are partly. If $f \in J_{k,\,m}$ is a cuspidal new eigenform, it generates an automorphic G^J-representation π of index m. But, for instance, the oldform $f \mid U_s$ (with $s \in \mathbb{N}$) generates a representation of index ms^2, which is certainly different from π. On the other hand, we have seen that Jacobi oldforms are not completely visible on the level of representations, since for example the linearly independent functions $f \mid U_s$ and $f \mid V_s^2$ generate the same representation.

To be a bit more specific, we make the following definition. Let F be a p-adic field and ω a prime element in F. A spherical representation π of $G^J(F)$ is called a *local newform*, if it is not a positive Weil representation, and if $V_{\omega^{-1}}\pi$ is not spherical (recall from Section 3 that the property of being spherical does only depend on the valuation of the index). The *degree* of a spherical representation is defined to be the dimension of the space of K^J-invariant vectors.

6.1. *Conjecture.* Let π be a local newform.

(i) The degree of $V_\omega^n\pi$ is $[(n+2)/2]$ for any $n \geqslant 0$. In particular, the degree of π is 1.

(ii) $V_s\pi$ is not spherical for any $s \in F^*$ with $v(s) < 0$.

By the results of [9] we know that this conjecture is true for $n = 0, 1, 2$ for the local newforms $\pi^J_{\chi,\,m}$ in the good case (χ an unramified character).

We will work now over $\mathbb{Q}$ to give some evidence for conjecture i). Recall the notation

$$\hat{G}^{J,0}_{k,m}$$

for the set of classical automorphic representations introduced in the previous section. We have not yet answered the following question: Can it happen that a Jacobi newform f and an oldform f', both assumed to be cuspidal eigenforms of the same index, generate the same automorphic representation? It can not: f and f' would have the same Hecke eigenvalues at almost every place. The same would then be true for the corresponding elliptic modular forms F and F' under the Skoruppa–Zagier map (see [13]). But then F and F' would certainly be multiples of each other.

An element $\pi \in \hat{G}^{J,0}_{k,m}$ is called a *newform* if it is not of the form $V_{m/m'}\pi'$ for some $\pi' \in \hat{G}^{J,0}_{k,m'}$ with a proper divisor m' of m. By Theorem 5.1 and the last observation the newforms are exactly those classical automorphic representations generated by classical Jacobi newforms. If conjecture (ii) is true, then it is also clear that the newforms are exactly those classical automorphic representations which have local newforms as local components at every prime. In any case, the following holds, as is easy to see from Theorem 5.1.

6.2. Proposition. *The local components of an automorphic G^J-representation generated by a cuspidal new Jacobi eigenform are local newforms.*

Let $\hat{G}^{J,0,\text{new}}_{k,m} \subset \hat{G}^{J,0}_{k,m}$ denote the set of newforms, and $\hat{G}^{J,0,\text{old}}_{k,m}$ its complement, so that

$$\hat{G}^{J,0}_{k,m} = \hat{G}^{J,0,\text{new}}_{k,m} \coprod \hat{G}^{J,0,\text{old}}_{k,m}.$$

By what we have said above, the elements of $\hat{G}^{J,0,\text{old}}_{k,m}$ are exactly the representations attached to Jacobi oldforms of index m. Taking conjecture (i) for granted, let us count the degrees of all the elements of $\hat{G}^{J,0,\text{old}}_{k,m}$. Here the degree is certainly the dimension of the space of global spherical vectors, and equals the product of all local degrees.

The analogue of formula (7) is

$$\hat{G}^{J,0,\text{old}}_{k,m} = \coprod_{m' \mid m,\, m' \neq m} V_{m/m'} \hat{G}^{J,0,\text{new}}_{k,m}.$$

Since everything is multiplicative, we restrict to the case $m = p^n$, where we have

$$\hat{G}^{J,0,\text{old}}_{k,p^n} = \coprod_{\alpha=0}^{n-1} V_{p^{n-\alpha}} \hat{G}^{J,0,\text{new}}_{k,p^\alpha}.$$

Assuming conjecture (i), it follows that

$$\text{total degree of } \hat{G}^{J,\,0,\,\text{old}}_{k,\,p^n} = \sum_{n=0}^{\alpha-1} (\text{total degree of } \hat{G}^{J,\,0,\,\text{new}}_{k,\,p^\alpha}) \cdot \left[\frac{n-\alpha+2}{2}\right].$$

By conjecture (i) and Proposition 6.2 it further follows that

$$\text{total degree of } \hat{G}^{J,\,0,\,\text{new}}_{p,\,p^\alpha} = \dim J^{\text{cusp, new}}_{k,\,p^\alpha}.$$

Hence our formula reads

$$\dim J^{\text{cusp, old}}_{k,\,p^n} = \sum_{n=0}^{\alpha-1} \dim J^{\text{cusp, new}}_{k,\,p^\alpha} \cdot \left[\frac{n-\alpha+2}{2}\right] \tag{8}$$

Now this is in fact the correct formula, which follows from (7). Thus we see that our conjecture (i) is exactly what we would need to reproduce the dimension formula (8).

7. THE "CERTAIN SPACE" OF SKORUPPA AND ZAGIER

This section collects some more evidence for Conjecture 6.1 by considering analogues of the operators U_d and V_d for classical elliptic modular forms. The main result (Theorem 7.6) remains completely within classical modular forms, and may be of some interest independently of the theory of Jacobi forms.

The same considerations that lead to Conjecture 6.1 could also be made in the case of classical elliptic modular forms. One would arrive at a conjecture about the dimensions of spaces of vectors in local representations invariant under congruence subgroups, and this conjecture would match perfectly with the results of Casselman in [3].

We recall this results here. Let F be a non-archimedean local field, and let $\mathcal{O}$, ω and v have their usual meanings. For every integer $n \geqslant 0$ let

$$K_0(\omega^n) = \left\{ \begin{pmatrix} a & b \\ c & c \end{pmatrix} \in \mathrm{GL}(2, \mathcal{O}) : v(c) \geqslant n \right\}$$

be a local congruence subgroup. Thus $K_0(\omega^0) = \mathrm{GL}(2, \mathcal{O})$. Let $(\pi, \mathcal{V})$ be an irreducible, admissible, infinite-dimensional representation of $\mathrm{GL}(2, F)$. For simplicity, we shall assume throughout that π *has trivial central character*, since this is sufficient for the applications we have in mind. For an integer $n \geqslant 0$ let $\mathscr{V}^n$ be the space of elements of $\mathscr{V}$ invariant under $K_0(\omega^n)$. Let $\mathscr{V}^{(-1)} = \{0\}$ Then it is known that there exists an integer $n \geqslant 0$ such that $\mathscr{V}^{(n)}$ is non-zero, but $\mathscr{V}^{(-1)} = \{0\}$ is zero. We call n the *conductor* of π. The following theorem is due to Casselman (see [3]).

7.1. THEOREM. *Let n be the conductor of the irreducible, admissible, infinite-dimensional representation $(\pi, \mathscr{V})$ of* GL$(2, F)$, *assumed to have trivial central character. Then*

$$\dim(\mathscr{V}^{(n+l)}) = l + 1 \qquad \textit{for every} \quad l \geqslant 0.$$

A non-zero element of $\mathscr{V}^{(n)}$, where n is the conductor, is thus unique up to scalars, and is called a *local newform.* For $n = 0$ the local newform is a spherical vector. It is obvious that the operator $\pi\left(\begin{smallmatrix} \omega^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ induces a linear map

$$\pi \begin{pmatrix} \omega^{-1} & 0 \\ 0 & 1 \end{pmatrix} : \mathscr{V}^{(n+l)} \to \mathscr{V}^{(n+l+1)}$$

for every $l \geqslant 0$. Another such map is the inclusion. We include a result which is also proved by Deligne in [4, Théorème 2.2.6].

7.2. PROPOSITION. *Let $(\pi, \mathscr{V})$ be an irreducible, admissible, infinite-dimensional representation of* GL$(2, F)$. *Let n be the conductor of π, and let $v \in \mathscr{V}$ be a local newform. For every integer $l \geqslant 0$, the vectors*

$$v_i := \pi \begin{pmatrix} \omega^{-1} & 0 \\ 0 & 1 \end{pmatrix} v, \qquad i = 0, ..., l \tag{9}$$

constitute a basis of $\mathscr{V}^{(n+l)}$.

Proof. The proof is by induction on l, the case $l = 0$ being trivial. Assume the assertion is true for l, but wrong for $l + 1$. Since obviously $v_{l+1} \in \mathscr{V}^{(n+l+1)}$, this would mean that $v_{l+1} \in \mathscr{V}^{(n+l)}$. By induction the space $\mathscr{V}^{(n+l)}$ is then invariant under $\left(\begin{smallmatrix} \omega^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and thus also under all matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $a \in F^*$. Since the same space is invariant under $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, it follows that it is also invariant under

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

hence under all of $N(F)$, where N is the unipotent radical of the standard Borel subgroup. Since $N(F)$ and any matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with non-vanishing c

generate $\mathrm{SL}(2, F)$, it follows that $\mathscr{V}^{(n+l)}$ is invariant under $\mathrm{SL}(2, F)$. Together with the invariance under matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we conclude that it is invariant under all of $\mathrm{GL}(2, F)$, an obvious contradiction. ∎

We continue to assume that π has trivial central character, i.e., we are dealing with a representation of $\mathrm{PGL}(2, F)$. We shall define the *local Atkin–Lehner involutions*, which are involutions on the finite-dimensional spaces $\mathscr{V}^{(n)}$ (now n is not necessarily the conductor). Let

$$A_n = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, F)$$

be any matrix with the property that

$$a, b, c, d \in \mathcal{O}, \qquad v(a) \geqslant n, \quad v(c) \geqslant n, \quad v(d) \geqslant n, \quad \det(A_n) = \varpi^n.$$

Such an A_n exists, e.g.,

$$A_n = \begin{pmatrix} \varpi^n & 1 \\ \varpi^n(\varpi^n - 1) & \varpi^n \end{pmatrix}.$$

The following facts are easily proved:

(i) A_n is unique up to right multiplication by elements of $K_0(\varpi^n)$.

(ii) A_n normalizes $K_0(\varpi^n)$.

(iii) $A_n^2 \in \varpi^n K_0(\varpi^n)$.

In view of (i), the effect of $\pi(A_n)$ on an element of $\mathscr{V}^{(n)}$ is well defined. Because of (ii), the result will again lie in $\mathscr{V}^{(n)}$. We denote by B_n the endomorphism of $\mathscr{V}^{(n)}$ thus defined. By (iii) and our assumption that π have trivial central character, B_n is an involution. This is the *local Atkin–Lehner involution of level n*.

7.3. PROPOSITION. *Let n be the conductor of the irreducible, admissible, infinite-dimensional representation* $(\pi, \mathscr{V})$ *of* $\mathrm{GL}(2, F)$. *Let v be a local newform for* π *and* $\varepsilon \in \{\pm 1\}$ *be defined by* $B_n v = \varepsilon v$. *Then for every* $l \geqslant 0$, *the matrix of the endomorphism* $B_{n+l}\colon \mathscr{V}^{(n+l)} \to \mathscr{V}^{(n+l)}$ *with respect to the basis* (9) *is the* $(l+1) \times (l+1)$*-matrix*

$$\begin{pmatrix} & & \varepsilon \\ & \cdot^{\cdot^{\cdot}} & \\ \varepsilon & & \end{pmatrix}.$$

Proof. The proof is by induction on l, the case $l=0$ being trivial. Fix an $l \geqslant 1$ and assume the assertion is true for $l-1$. Then for any $i \in \{1, ..., l\}$ we have, with v_i as in Proposition 7.2,

$$\begin{aligned} B_{n+l} v_i &= \pi \begin{pmatrix} \omega^{n+l} & 1 \\ \omega^{n+l}(\omega^{n+l}-1) & \omega^{n+l} \end{pmatrix} \pi \begin{pmatrix} \omega^{-1} & 0 \\ 0 & 1 \end{pmatrix} v_{i-1} \\ &= \pi \begin{pmatrix} \omega^{n+l-1} & 1 \\ \omega^{n+l-1}(\omega^{n+l}-1) & \omega^{n+l} \end{pmatrix} v_{i-1}. \end{aligned}$$

This last matrix is a version of A_{n+l-1}. Hence by the induction hypothesis

$$B_{n+l} v_i = B_{n+l-1} v_{i-1} = \varepsilon v_{l-i} \qquad \text{for} \quad i \in \{1, ..., l\}. \tag{10}$$

It remains to prove that this equation also holds for $i=0$. Let

$$B_{n+l} v_0 = \sum_{i=0}^{l} \alpha_i v_i \tag{11}$$

with complex constants α_i. Since the matrix of the involution B_{n+l} is non-singular, we must have $\alpha_l \neq 0$. Applying to (11) the operator B_{n+l} and using (10), we obtain

$$\begin{aligned} v_0 &= \varepsilon \sum_{i=1}^{l} \alpha_i v_{l-i} + \alpha_0 \sum_{i=0}^{l} \alpha_i v_i \\ &= \sum_{i=0}^{l-1} (\varepsilon \alpha_{l-i} + \alpha_0 \alpha_i)\, v_i + \alpha_0 \alpha_l v_l. \end{aligned}$$

It follows that $\alpha_0 \alpha_l = 0$, and since $\alpha_l \neq 0$, that $\alpha_0 = 0$. Then it further follows that

$$v_0 = \varepsilon \sum_{i=0}^{l-1} \alpha_{l-i} v_i,$$

and one sees that $\alpha_l = \varepsilon$ and $\alpha_1 = \cdots = \alpha_{l-1} = 0$. ∎

7.4. Corollary. *Let $\mathscr{V}_{\pm}^{(n+l)}$ be the ± 1-eigenspace of the involution B_{n+l} on $\mathscr{V}^{(n+l)}$. Then*

$$\dim \mathscr{V}_{\varepsilon}^{(n+l)} = \left[\frac{l+2}{2}\right], \qquad \dim \mathscr{V}_{-\varepsilon}^{(n+l)} = \left[\frac{l+1}{2}\right].$$

We go one step further and define operators

$$U\colon \mathcal{V}^{(n+l)} \to \mathcal{V}^{(n+l+2)},$$
$$w \mapsto \pi\begin{pmatrix} \omega^{-1} & 0 \\ 0 & 1 \end{pmatrix} w, \tag{12}$$

and

$$V\colon \mathcal{V}^{(n+l)} \to \mathcal{V}^{(n+l+1)},$$
$$w \mapsto w + \pi\begin{pmatrix} \omega^{-1} & 0 \\ 0 & 1 \end{pmatrix} w. \tag{13}$$

Note that we do not index these operators by l or $n+l$. It is clear that U and V commute.

7.5. Corollary. *If v is a local newform with $B_n v = \varepsilon v$, then for any $l \geqslant 0$, the space $\mathcal{V}_{\varepsilon}^{(n+l)}$ has a basis consisting of the vectors*

$$U^d V^{d'} v, \qquad \textit{where} \quad d, d' \geqslant 0, \quad 2d + d' = l.$$

In other words,

$$\mathcal{V}_{\varepsilon}^{(n+l)} = \bigoplus_{\substack{d, d' \geqslant 0 \\ 2d + d' = l}} U^d V^{d'} \mathcal{V}^{(n)}.$$

Proof. If $B_{n+l} w = \varepsilon w$ for some $w \in \mathcal{V}^{(n+l)}$, then

$$B_{n+l+2}(Uw) = \varepsilon(Uw) \qquad \text{and} \qquad B_{n+l+1}(Vw) = \varepsilon(Vw).$$

This follows by the equations

$$B_{n+l+1}\pi\begin{pmatrix} \omega^{-1} & 0 \\ 0 & 1 \end{pmatrix} w = B_{n+l} w = \varepsilon w, \qquad B_{n+l+1} w = \varepsilon\pi\begin{pmatrix} \omega^{-1} & 0 \\ 0 & 1 \end{pmatrix} w$$

(the second follows from the first). Thus U (resp. V) maps $\mathcal{V}_{\varepsilon}^{(n+l)}$ to $\mathcal{V}_{\varepsilon}^{(n+l+2)}$ (resp. $\mathcal{V}_{\varepsilon}^{(n+l+1)}$). There are exactly $[(l+2)/2]$ pairs (d, d') such that $2d + D' = l$. By Corollary 7.4, it remains only to prove that the vectors $U^d V^{d'} v$ are linearly independent. But knowing the linear independence of the vectors (9), this is easily seen. ∎

Now we turn to global considerations in the classical context. Let

$$f \in \mathscr{S}_k(\Gamma_0(m))$$

be any eigenform, and let $\pi = \otimes \pi_p$ be the automorphic GL(2)-representation generated by f. Let f' be the newform from which f "comes from", meaning f' is a newform of the same weight as f and of a level $m' \mid m$, and for almost all primes p the modular forms f and f' share the same Hecke eigenvalue. This f' is unique up to scalars, and also lies in the space of π. Indeed, f' is obtained by piecing together the local newforms of the representations π_p for every finite p, and the lowest weight vector in π_∞. Write

$$m' = \prod p^{n_p}, \qquad m = \prod p^{n_p + l_p}.$$

Then n_p is the conductor of the local representation $(\pi_p, \mathscr{V}_p)$, and the (adelic) function f is obtained by piecing together local vectors contained in the spaces $\mathscr{V}_p^{(n_p + l_p)}$.

The newform f' is automatically an eigenform for all Atkin–Lehner involutions W_p, $p \mid m$ (if $p \mid m$ but $p \nmid m'$, then W_p is the identity). Let $\varepsilon_p \in \{\pm 1\}$ be the eigenvalue. Now assume that f is also an eigenform for all W_p, $p \mid m$. This is the case if and only if f is obtained by piecing together local vectors contained in $\mathscr{V}_{p,+}^{(n_p+l_p)}$, or in $\mathscr{V}_{p,-}^{(n_p+l_p)}$. By the definitions, f will have the same eigenvalue ε_p as f' under the Atkin–Lehner involution at p if and only if the local component of f at p comes from the space $\mathscr{V}_{p,\varepsilon_p}^{(n_p+l_p)}$. Consider the subspace of $\mathscr{S}_k(\Gamma_0(m))$ spanned by all modular forms f with exactly this property. This is the "certain space" in the title of [13]. As in this paper we denote it by

$$\mathfrak{S}_k(m).$$

Skoruppa and Zagier studied this space because $\mathfrak{S}_{2k-2}(m)$ contains the image of the Hecke-equivariant embedding

$$J_{k,m}^{\mathrm{cusp}} \to \mathscr{S}_{2k-2}(\Gamma_0(m)),$$

the existence of which is one of the main results of [13]. More precisely, let $\mathscr{S}_{2k-2}^{-}(\Gamma_0(m))$ be the subspace of elements $f \in \mathscr{S}_{2k-2}(\Gamma_0(m))$ satisfying $f(-1/m\tau) = (-1)^k m^{k-1} \tau^{2k-2} f(\tau)$, and let $\mathfrak{S}_{2k-2}^{-}(m) = \mathfrak{S}_{2k-2}(m) \cap \mathscr{S}_{2k-2}^{-}(\Gamma_0(m))$. Then it is proved in [13] that there exists a Hecke-equivariant isomorphism

$$J_{k,m}^{\mathrm{cusp}} \to \mathfrak{S}_{2k-2}^{-}(m). \tag{14}$$

(One can extend this map by considering the space $J^{*}_{k,m}$ of *skew-holomorphic* Jacobi forms, cf. [1, 4.1]. It can be shown that there exists a Hecke-equivariant isomorphism

$$J^{\mathrm{cusp}}_{k,m} \oplus J^{*,\,\mathrm{cusp}}_{k,m} \to \mathfrak{S}_{2k-2}(m);$$

see [12]. We will content ourselves with the "incomplete" version (14).)

Now we define global analogues of the local operators (12) and (13). Let $f \in \mathscr{S}_k(\Gamma_0(m))$ be any eigenform, considered as an adelic function. Assume that $f = \prod f_p$ with respect to the decomposition $\pi_f = \otimes \pi_p$, where f_p lies in any fixed model $\mathscr{V}_p$ of π_p. If $m = \prod p^{n_p}$, then f_p lies in $\mathscr{V}_p^{(n_p)}$. Now fix any p, and apply to f_p the local operator

$$U_p : \mathscr{V}_p^{(n_p)} \to \mathscr{V}_p^{(n_p+2)}$$

defined in (12). The resulting function

$$f \mid \mathscr{U}_p := \prod_{p' \neq p} f_{p'} \times U_p f_p$$

is an element of $\mathscr{S}_k(\Gamma_0(mp^2))$. It is easy to see that $\mathscr{U}_p f(z) = f(pz)$ when f and $\mathscr{U}_p f$ are considered as classical functions on the upper half plane. This leads us to define the operator

$$\begin{aligned} \mathscr{U}_p : \mathscr{S}_k(\Gamma_0(m)) &\to \mathscr{S}_k(\Gamma_0(mp^2)), \\ f &\mapsto (z \mapsto f(pz)). \end{aligned} \tag{15}$$

This is certainly the well-known operator producing oldforms from newforms, but we consider it as an operator multiplying the level by p^2, not only by p. By Corollary 7.5 this makes $\mathscr{U}_p$ preserve Atkin–Lehner eigenvalues.

Very similar considerations can be made with the local operator V defined in (13), and these lead to a consistent definition of a global operator

$$\begin{aligned} \mathscr{V}_p : \mathscr{S}_k(\Gamma_0(m)) &\to \mathscr{S}_k(\Gamma_0(mp)), \\ f &\mapsto (z \mapsto f(z) + f(pz)), \end{aligned}$$

which also preserves Atkin–Lehner eigenvalues.

We extend the definition of $\mathscr{U}_p$ and $\mathscr{V}_p$ to an arbitrary positive integer $d = \prod p^{\alpha_p}$ by setting

$$\mathscr{U}_d = \prod \mathscr{U}_p^{\alpha_p}, \qquad \mathscr{V}_d = \prod \mathscr{V}_p^{\alpha_p}.$$

Then $\mathscr{U}_d$ multiplies the level by d^2, and $\mathscr{V}_d$ multiplies the level by d. The operator $\mathscr{U}_d$ simply sends $f(z)$ to the function $f(dz)$, while $\mathscr{V}_d$ can not be described in such a simple manner.

The following theorem is an immediate consequence of the definitions and the local considerations above, in particular Corollary 7.5.

7.6. THEOREM. *The Skoruppa–Zagier space $\mathfrak{S}_k(m)$ may be described as*

$$\mathfrak{S}_k(m) = \bigoplus_{\substack{l,\, l' \in \mathbb{N} \\ l^2 l' \mid m}} \mathscr{S}_k(\Gamma_0(m/l^2 l'))^{\mathrm{new}} \mid \mathscr{U}_l \mathscr{V}_{l'}.$$

This is the analogue of formula (7) for Jacobi forms. Since the operators $\mathscr{U}_d$ and $\mathscr{V}_d$ were designed to preserve Hecke eigenvalues for all W_p and almost all $T(p)$, and their analogues U_d and V_d for Jacobi forms do also, it follows that once one has defined for every level $m \in \mathbb{N}$ a Hecke-equivariant embedding

$$\mathscr{S}: J_{k,m}^{\mathrm{cusp,\, new}} \to \mathscr{S}_{2k-2}(\Gamma_0(m))^{\mathrm{new}}, \tag{17}$$

this may be extended to a Hecke-equivariant embedding

$$J_{k,m}^{\mathrm{cusp}} \to \mathscr{S}_{2k-2}(\Gamma_0(m)), \tag{18}$$

by simply sending $f \mid U_l V_{l'}$ to $(\mathscr{S}f) \mid \mathscr{U}_l \mathscr{V}_{l'}$. The image of (18) will lie in $\mathfrak{S}_{2k-2}(m)$ (and will in fact equal $\mathfrak{S}_{2k-2}^{-}(m)$). The hard thing is to prove the existence of the map (17), which is done in [13]. (As mentioned before, these results can be improved by including skew-holomorphic Jacobi forms. The resulting embeddings are then onto).

8. AN APPLICATION

As an application of index shifting we will now determine the local components of automorphic representations attached to Jacobi forms of square free index.

In the following we will make use of the classification of the spherical representations of G^J in the good and almost good case. We thus recall the results of [10]. Let F be a $\mathfrak{p}$-adic field with odd residue characteristic, or let $F = \mathbb{Q}_2$. Let ψ be a character of F with conductor $\mathcal{O}$, the ring of integers of F. Recall that for an irreducible, admissible representation π of $G^J = G^J(F)$ of index $m \in F^*$ we say that we are in the *good case*, if $v(m) = 0$, or in the *almost good case*, if $v(m) = 1$. In these cases the Jacobi Hecke algebras (defined in Section 3) are commutative, and consequently

TABLE I

Representation	$T^J(\omega)$-eigenvalue	W-eigenvalue
	Good case	
$\pi^J_{\chi,m}$ with χ unramified	$q^{3/2}(\chi(\omega)+\chi(\omega)^{-1})$	1
$\sigma^{J+}_{1,m}$	$q(q+1)$	1
$\sigma^{J+}_{\xi,m}$	$-q(q-1)$	1
	Almost good case	
$\pi^J_{\chi,m}$ with χ unramified	$q^{3/2}(\chi(\omega)+\chi(\omega)^{-1})+q(q-1)$	1
$\sigma^{J+}_{1,m}$	$2q^2$	1
$\sigma^{J}_{\xi,m}$	$-2q$	1
$\sigma^{J-}_{1,m}$	0	-1

the degree of any spherical representation is 1. Table I is a complete list of all spherical representations in the good and almost good case. The element $\xi \in F^*$ which appears is any element of $\mathcal{O}^* \backslash \mathcal{O}^{*2}$ if the residue characteristic of F is odd, resp. 5 if $F = \mathbb{Q}_2$.

The last column of Table I gives the eigenvalue of a spherical vector under the Heisenberg involution (in the good case, this is not really an involution, but the identity). The information in the middle column is not really necessary for our purposes.

8.1. Proposition. *Let $f \in J^{\text{cusp}}_{k,m}$ be a new eigenform of square free index, and $\pi = \otimes \pi_p$ the associated automorphic G^J-representation.*

(i) *The archimedean component of π is $\pi_\infty = \pi^{J+}_{m,k}$*

(ii) *For $p \nmid m$, the component π_p is a principal series representation.*

(iii) *For $p \mid m$, we have $\pi_p = \sigma^J_{\xi,m}$ if $W_p f = f$, and $\pi_p = \sigma^{J-}_{1,m}$ if $W_p f = -f$.*

Proof. For statements (i) and (ii) we refer to [1, 7.5]. Suppose $p \mid m$. Once we know that π_p is a special or a negative Weil representation, the assertions in (iii) follow from the Heisenberg eigenvalues given in the above table. Hence assuming that (iii) is false, we are left with the possibilities

$$\pi_p = \pi^J_{\chi,m} \text{ with unramified } \chi, \qquad \text{or} \qquad \pi_p = \pi^{J+}_{1,m}.$$

(As was mentioned earlier, we know that positive Weil representations can not occur, but we do not need this result here.) Now apply the global index shifting operator $V_{p^{-1}}$ to π. By the above table and Proposition 2.3, the result is a classical automorphic representation π' of index mp^{-1}. Let f' be the Jacobi form contained in π' (unique up to scalars, since we are in the good and almost good case at all places). By Theorem 5.1, the Jacobi forms f and $f \mid V_p$ and the representations π' and π make a commutative diagram. Since all local representations are of degree 1, we conclude that f is a multiple of $f' \mid V_p$. This is a contradiction to f being a newform. ∎

We could have proved this result faster by using Proposition 6.2, but we wanted to avoid making implicit use of the Skoruppa–Zagier correspondence. There is another proof of Proposition 8.1 which uses the analogous result for elliptic cusp forms (see [7]) and the correspondence between the Jacobi group and GL(2) (cf. [11, 8.3]). However, the proof we gave here has the advantage of being purely Jacobi theoretic. In turn we can then deduce the corresponding results for elliptic modular forms.

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